Question 0.1. *Proof that $\sqrt{1}+\sqrt{2}+\sqrt{3}\cdots+\sqrt{n}$ for $n>2$ is irrational.*

However, if we consider the linear combination of $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}\ldots$ $\sqrt{n}$ over $\mathbb{Q}$, then it is likely that we will get a similar result as we kick out some special cases.

Question 0.2. *Given n positive integers, $a_1 < a_2 < \cdots < a_n$, such that $\sqrt{\frac{a_j}{a_i}} \notin \mathbb{Q}$ for all $i<j$ and $\sqrt{a_i} \notin \mathbb{Q}$ for all i. Also given $\lambda_1, \lambda_2 \ldots \lambda_n \in \mathbb{Q}$. Prove that $\sum_{i=1}^{n} \lambda_i \sqrt{a_i} \in \mathbb{Q}$ if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.*

Proof. Suppose this stand for $n=k$, then it is suffice to prove $n=k+1$.

Suppose there exist $\lambda_1, \lambda_2 \ldots \lambda_{k+1} \in \mathbb{Q}$ such that

$$\sum_{i=1}^{k+1} \lambda_i \sqrt{a_i} = q \in \mathbb{Q}$$

If $\lambda_j = 0$, then $-\lambda_j\sqrt{a_j} + \sum_{i=1}^{k+1} \lambda_i \sqrt{a_i} = q \in \mathbb{Q}$, and we are done according to the assumption.

If $\lambda_j \neq 0$ for all j, then we have

$$\sum_{i=1}^{k} \lambda_i \sqrt{a_i} = q - \lambda_{k+1}\sqrt{a_{k+1}}$$

Clearly, $q - \lambda_{k+1}\sqrt{a_{k+1}}$ is a root of a irreducible quadratic polynomial $g \in \mathbb{Q}[x]$. And the polynomial must have another root which is $q + \lambda_{k+1}a_{k+1}$.

Also, the polynomial

$$f = \prod (x - (\pm\lambda_1\sqrt{a_1} + \pm\lambda_2\sqrt{a_2} + \cdots + \pm\lambda_n\sqrt{a_n}))$$

is also in $\mathbb{Q}[x]$ by induction.

Also,

$$\begin{aligned} f(q - \lambda_{k+1}\sqrt{a_{k+1}}) &= f(\sum_{i=1}^{k+1} \lambda_i \sqrt{a_i}) \\ &= 0 \end{aligned}$$

As g is irreducible over $\mathbb{Q}$, g must divides f. Thus

$$f(q + \lambda_{k+1}\sqrt{a_{k+1}}) = 0$$

Thus, there exits $\alpha_1, \alpha_2 \ldots \alpha_n$ which is either 1 or -1 such that

$$\sum_{i=1}^{k} \alpha_i\lambda_i \sqrt{a_i} = q + \lambda_{k+1}\sqrt{a_{k+1}}$$

Thus

$$\begin{cases} \sum_{i=1}^{k} \alpha_i\lambda_i \sqrt{a_i} = q + \lambda_{k+1}\sqrt{a_{k+1}} \\ \sum_{i=1}^{k} \lambda_i \sqrt{a_i} = q - \lambda_{k+1}\sqrt{a_{k+1}} \end{cases}$$

$$\begin{aligned} \sum_{i=1}^{k} \alpha_i\lambda_i \sqrt{a_i} + \sum_{i=1}^{k} \lambda_i \sqrt{a_i} &= q + \lambda_{k+1}\sqrt{a_{k+1}} + q - \lambda_{k+1}\sqrt{a_{k+1}} \\ \sum_{i=1}^{k} (\alpha_i + 1)\lambda_i \sqrt{a_i} &= 2q \in \mathbb{Q} \end{aligned}$$

By the induction assumption, we have $(\alpha_i + 1)\lambda_i = 0$ for all $i \leq k$. As $\lambda_i \neq 0$ for all i, we have $\alpha_i = -1$ for all $i \leq k$.

Thus,

$$\sum_{i=1}^{k}(\alpha_i + 1)\lambda_i\sqrt{a_i} \quad = \quad 0$$

which implies $q = 0$.

Thus,

$$\begin{aligned}
\sum_{i=1}^{k+1}\lambda_i\sqrt{a_i} &= 0 \\
\sqrt{a_{k+1}}(\sum_{i=1}^{k+1}\lambda_i\sqrt{a_i}) &= 0 \\
\sum_{i=1}^{k}\lambda_i\sqrt{a_{k+1}a_i} &= -\lambda_{k+1}a_{k+1} \in \mathbb{Q}
\end{aligned}$$

The only thing left to proof is that $a_{k+1}a_i, i \leq k$ satisfy the assumption in the question, and then the induction assumption will apply, which proves that $\lambda_i = 0, i \leq k$.

Given any $i < j$, we have

$$\begin{aligned}
\sqrt{\frac{a_j a_{k+1}}{a_i a_{k+1}}} &= \sqrt{\frac{a_j}{a_i}} \notin \mathbb{Q} \\
\sqrt{a_j a_{k+1}} &= a_{k+1}\sqrt{\frac{a_j}{a_{k+1}}} \notin \mathbb{Q}
\end{aligned}$$

□

As, we does not use any special property of $\mathbb{Q}$ other then the fact that it is a field. So, the proof should be valid for any field $\mathbb{F}$ that characteristic equals 0, if we rephrase the question in term of $\mathbb{F}$.

Question 0.3. *Given n distinct element $a_1, a_2, \ldots, a_n \in \mathbb{F}$, let b_i be a root of $x^2 - a_i$ for all i. And $b_i b_j^{-1} \notin \mathbb{F}$ for all $i \neq j$, and $b_i \notin \mathbb{F}$ for all i. Then given $\lambda_1, \lambda_2 \ldots \lambda_n \in \mathbb{F}$, $\sum_{i=0}^{n}\lambda_i b_i \in \mathbb{F}$ if and only if $\lambda_i = 0$ for all i.*

If we try to shift b_k to $b_k + c_k$ such that $c_k \in \mathbb{F}$, then the result clearly still stand. So, the previous question can be generalised a bit.

Question 0.4. *Given n distinct irreducible quadratic polynomial $f_1, f_2 \ldots f_n$ in $\mathbb{F}[x]$.*

And given any $i \neq j$, f_j does not have root in $\mathbb{F}[x]/f_i$.

And b_i be all roots of f_i for all i. Then given $\lambda_1, \lambda_2 \ldots \lambda_n \in \mathbb{F}$, $\sum_{i=0}^{n}\lambda_i b_i \in \mathbb{F}$ if and only if $\lambda_i = 0$ for all i.

If we are not satisfied with the quadratic polynomials, we can try to generalise the question to any polynomial. However, we may need to have a more strict condition on the roots.

Question 0.5. *Given n distinct irreducible polynomial $f_1, f_2 \dots f_n$ in $\mathbb{F}[x]$ with orders greater or equal than 2.*

Given any $i \neq j$. f_j does not have root in $\mathbb{F}[x]/f_i$.

And b_i be all roots of f_i for all i. Then given $\lambda_1, \lambda_2 \dots \lambda_n \in \mathbb{F}$, $\sum_{i=0}^{n} \lambda_i b_{i,1} \in \mathbb{F}$ if and only if $\lambda_i = 0$ for all i.

To prove the above question, we need another definition and some relating lemma.

Definition 0.1. *Given a polynomial $f(x) \in \mathbb{F}[x]$,*

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

Define the derivative $f'(x)$ of $f(x)$ as

$$f'(x) = nx^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + 2a_{n-2}x + a_{n-1}$$

Through some simple calculation, we can prove the following lemma.

Lemma 0.1. *Given polynomials $f(x), g(x) \in \mathbb{F}[x]$,*

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Lemma 0.2. *Given a irreducible polynomial $f(x) \in \mathbb{F}[x]$, and a field $\mathbb{K}$, such that $\mathbb{F} \subseteq \mathbb{K}$ and $f(x)$ can be factorized into product of linear factors in $\mathbb{K}$.*

Chose a root α of $f(x)$ in $\mathbb{K}$. Then $(x-\alpha)^2$ divides $f(x)$ in $\mathbb{K}$, if and only if $f'(x) = 0$.

As $f'(x)$ is still a polynomial in $\mathbb{F}[x]$, and we can choose arbitrary $\mathbb{K}$. This lemma implies that irreducible polynomial $f(x)$ have distinct roots if and only if $f'(x) \neq 0$.

Proof. If $(x-\alpha)^2$ divides $f(x)$ in $\mathbb{K}$.

Let,

$$f(x) = (x-\alpha)^2 g(x)$$

Then,

$$f'(x) = 2(x-\alpha)g(x) + (x-\alpha)^2 g'(x)$$

Then $f'(\alpha) = 0$. As $f(x)$ is irreducible, and $f(x)$ and $f'(x)$ have common root. So, $f(x)$ divides $f'(x)$.

As, the degree of $f'(x)$ is less than $f(x)$, then $f'(x) = 0$.

Conversely, if $f'(x) = 0$.

Let,

$$f(x) = (x-\alpha)g(x)$$

Then,

$$f'(x) = g(x) + (x-\alpha)g'(x)$$

Then,

$$\begin{aligned} 0 &= f'(\alpha) \\ &= g(\alpha) + (\alpha-\alpha)g'(\alpha) \\ &= g(\alpha) \end{aligned}$$

So, α is a root of $g(x)$.

Thus, $(x-\alpha)^2$ divides $f(x)$ in $\mathbb{K}$. □

Lemma 0.3. *Given distinct irreducible polynomials $f(x), g(x) \in \mathbb{F}[x]$, where the characteristic of $\mathbb{F}$ equals 0.*

Chose any field $\mathbb{K}$, such that $\mathbb{F} \subseteq \mathbb{K}$, and $f(x)$ and $g(x)$ can be factorized into product of linear factors in $\mathbb{K}$.

Let $a_1, a_2 \ldots a_n$ be all roots of $f(x)$ in $\mathbb{K}$, and $b_1, b_2 \ldots b_m$ be all roots of $g(x)$ in $\mathbb{K}$.

Then,

$$h(x) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x - (a_i + b_j))$$

have coefficients in $\mathbb{F}$,